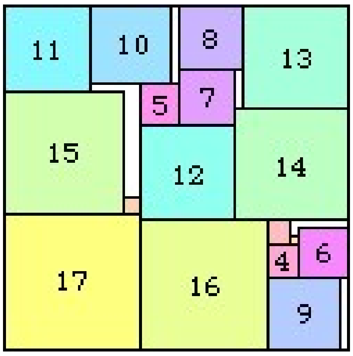
Task:

Let us consider the following problem.

Given the 1x1 square, the 2x2 square, and so on, finally the n x n square.
Which is the smallest K*K square in which the above can fit (parallel to the sides of the large square and without overlapping)?

Solution

The problem can be easily solved for small n numbers, for example, here we can see a solution for n=17.



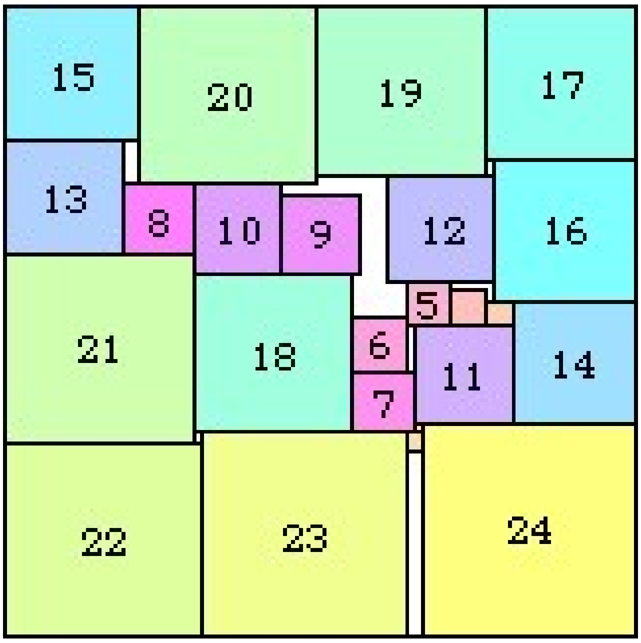
In the case of N=17, the situation is easy, since the total area of the small squares is 1785, the root of which is roughly 42.24, rounding this up we get 43, it is easy to see that such a large enclosing square is definitely needed. Since we managed to pack (somehow, even with luck) the small squares into such a big accomodating square, we "were clever", we can rest assured that this is an optimal packing. Then 43*43-1785=64 cells remained empty. (Source of the figure, Internet, Erich Friedman's website.)

However, for certain numbers n, the question is very difficult.

Already Watson (Watson, 1918) realized that n=24 is the only n (except for the trivial n=1), when the sum of the square numbers from 1 to n is itself a square number. If we add the square numbers from 1 to n=24, it is exactly 70*70. So the question arises (Gardner, 1966, 1975) whether these small squares fit into a large 70*70 square, without rotation or overlapping.

The answer is: no. Korf (Korf, 2003, 2004) gave a computer-aided proof for the statement, but a purely combinatorial proof (not using the help of a computer) had not been created before.

The members of the research group (jointly with other researchers: Sgall 2024) managed to come up with the first purely combinatoric proof. Here is another picture of packing in a 71*71 square (which is not difficult to get).



By definition, we cannot show a diagram when the squares do not fit into a large 70*70 square. But anyone can try this at home, which we guarantee will not succeed. However, this should not discourage us: what cannot be done cannot be done. But: we finally got an answer in the article about why it can't be.

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